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The uniformizing differential equation of the complex hyperbolic structure on the moduli space of a marked cubic surface: II

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Abstract
The relation between the uniformizing equation of the complex hyperbolic structure on the moduli space of marked cubic surfaces and an Appell–Lauricella hypergeometric system in nine variables is clarified.

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1. Introduction

In the previous paper [3] we found the uniformizing equation which governs the developing map of the complex hyperbolic structure on the (four-dimensional) moduli space of marked cubic surfaces. Our equation is invariant under the action of the Weyl group of type E_6. Terasoma and Matsumoto are establishing a theory implying that the variation of Hodge structure of the cubic surfaces is essentially equivalent to that of a certain family of curves which are cyclic covers of the line branching at 12 points [2]. This theory suggests that our uniformizing equation should be equivalent to a restriction of the Appell–Lauricella hypergeometric system F_D in nine variables. In this paper, we carry out the computation and prove this observation.

2. Configuration spaces $X(2, N)$

Let $X(2, N)$ be the configuration space of N (coloured) points on the projective line P^1 defined as

$$X(2, N) = PGL_2\backslash\{(x_1, \ldots, x_N) \in (\mathrm{P}^1)^N | x_i \neq x_j \ (i \neq j)\}.$$

By normalizing three points as $0, 1, \infty$, the space $X(2, N)$ can be identified with the open affine set

$$\prod_{j=1}^{n} x_j(1 - x_j) \prod_{1\leqslant i<j\leqslant n} (x_i - x_j) \neq 0$$

in the affine space coordinatized by $(x_1, \ldots, x_n)$, where $n + 3 = N$. On this configuration space lives the Appell–Lauricella hypergeometric system E_D, which we review in the next section. When $n = 1$, $X(2, 4)$ is isomorphic to $\mathbf{C} - \{0, 1\}$.

3. Appell–Lauricella hypergeometric system E_D

For coordinates $x_1, \dots, x_n$, put $D_i := x_i\partial/\partial x_i$. The system

$$D_iD_ju = \sum_{k=1}^{n} p_{ij}^k D_k u + p_{ij}^0 u \qquad (i, j = 1, \dots, n)$$

with parameters $(a, b, c) = (a, b_1, \dots, b_n, c)$, where

$$p_{ij}^k = p_{ji}^k \qquad p_{ij}^0 = p_{ji}^0 \qquad 1 \leqslant i, j, k \leqslant n \qquad p_{ij}^0 p_{ij}^k = 0 \qquad i \neq j \neq k \neq i$$

$$p_{ij}^i = b_j \frac{x_j}{x_i - x_j} \qquad i \neq j \qquad p_{ii}^0 = ab_i \frac{x_i}{1 - x_i}$$

$$p_{ii}^k = b_i \left(\frac{x_i}{1 - x_i} - \frac{x_i}{x_k - x_i} \right) \qquad i \neq k$$

$$p_{ii}^i = -\sum_{k \neq i} b_k \frac{x_k}{x_i - x_k} + \frac{(a + b_i)x_i - c + 1}{1 - x_i}$$

is called the Appell–Lauricella hypergeometric system of type D and is denoted by $E_D^n(a, b_1, \dots, b_n, c)$. The Appell–Lauricella hypergeometric series

$$F_D(a; b_1, \dots, b_n; c|x^1, \dots, x^n) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a, m_1 + \dots + m_n)(b_1, m_1) \cdots (b_n, m_n)}{(c, m_1 + \dots + m_n)m_1! \cdots m_n!} (x^1)^{m_1} \cdots (x^n)^{m_n}$$

where $(a, m) = a(a + 1) \cdots (a + m - 1)$, solves this system around the origin [1, 4]. This system has singularities along the divisor defined by

$$\prod_{j=1}^{n} x_j(1 - x_j) \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j) = 0$$

and at infinity, that is, this system is regular in the configuration space $X(2, n + 3)$. The rank (the dimension of the space of local solutions at a (any) regular point) is $n + 1$. When $n = 1$, $E_D^1(a, b, c)$ is the Gauss hypergeometric equation.

Note in general that if a system of the above form is of rank $n + 1$ then the coefficients p_{ij}^0 can be expressed in terms of p_{pq}^r $(1 \leqslant p, q, r \leqslant n)$ and their derivatives; so we often describe a system by presenting p_{ij}^k only.

4. The pull-back of E_D under an embedding of $X(2, n + 3)$ into $X(2, 2n + 4)$

Let us embed (a Zariski open subset of) the space $X(2, n + 3)$ into $X(2, 2n + 4)$ as

$$\iota : X(2, n + 3) \ni (0, \infty, 1, x_1, \dots, x_n)$$
$$\longmapsto (0, \infty, 1, x_1, \dots, x_n, -x_1, \dots, -x_n, -1) \in X(2, 2n + 4).$$

The space $X(2, 2n + 4)$ is of dimension $2n + 1$ and the Appell–Lauricella hypergeometric system E_D^{2n+1} is of rank $2n + 2$.

For a solution $u = u(x_1, \dots, x_{2n+1})$ of $E_D^{2n+1} = E_D^{2n+1}(a, b_1, \dots, b_{2n+1}, c)$, put

$$v(x_1, \dots, x_n) := u(x_1, \dots, x_n, -x_1, \dots, -x_n, -1).$$

For generic parameters (a, b, c), the system satisfied by the pull-back v of u under ι would be of rank $2n + 2$. We study whether the rank of this system may be of rank $n + 1$. We have

$$D_i v = (D_i + D_{i+n})u \qquad 1 \leqslant i \leqslant n$$

$$\begin{aligned} D_i D_j v &= (D_i D_j + D_i D_{j+n} + D_{i+n} D_j + D_{i+n} D_{j+n})u \\ &= \sum_{k=0}^{2n+1} (p^k_{i,j} + p^k_{i,j+n} + p^k_{i+n,j} + p^k_{i+n,j+n}) D_k u \end{aligned}$$

where $D_0 u := u$, and the right-hand sides denote the values restricted to

$$x_{n+1} = -x_1, \ldots, x_{2n} = -x_n \qquad x_{2n+1} = -1.$$

For $1 \leqslant i \neq j \leqslant n$, we have

$$\begin{aligned} D_i D_j v &= \sum_{k=1}^{2n} (p^k_{i,j} + p^k_{i,j+n} + p^k_{i+n,j} + p^k_{i+n,j+n}) D_k u \\ &= (p^i_{i,j} + p^i_{i,j+n}) D_i u + (p^{i+n}_{i+n,j} + p^{i+n}_{i+n,j+n}) D_{i+n} u \\ &\quad + (p^j_{i,j} + p^j_{i+n,j}) D_j u + (p^{j+n}_{i,j+n} + p^{j+n}_{i+n,j+n}) D_{j+n} u \end{aligned}$$

and

$$p^i_{i,j} + p^i_{i,j+n} = b_j \frac{x_j}{x_i - x_j} + b_{j+n} \frac{x_{j+n}}{x_i - x_{j+n}} = b_j \frac{x_j}{x_i - x_j} + b_{j+n} \frac{-x_j}{x_i + x_j}$$

$$p^{i+n}_{i+n,j} + p^{i+n}_{i+n,j+n} = b_j \frac{x_j}{x_{i+n} - x_j} + b_{j+n} \frac{x_{j+n}}{x_{i+n} - x_{j+n}} = b_j \frac{x_j}{-x_i - x_j} + b_{j+n} \frac{-x_j}{-x_i + x_j}.$$

When $i = j$, we have

$$\begin{aligned} D_i^2 v &= \sum_{k=0}^{2n+1} (p^k_{i,i} + 2p^k_{i,i+n} + p^k_{i+n,i+n}) D_k u \\ &= (p^i_{i,i} + 2p^i_{i,i+n} + p^i_{i+n,i+n}) D_i u + (p^{i+n}_{i,i} + 2p^{i+n}_{i,i+n} + p^{i+n}_{i+n,i+n}) D_{i+n} u \\ &\quad + \sum_{1 \leqslant k \neq i \leqslant n} \left[(p^k_{i,i} + p^k_{i+n,i+n}) D_k u + (p^{k+n}_{i,i} + p^{k+n}_{i+n,i+n}) D_{k+n} u \right] \\ &\quad + (p^{2n+1}_{i,i} + p^{2n+1}_{i+n,i+n}) D_{2n+1} u + (p^0_{i,i} + p^0_{i+n,i+n}) u \end{aligned}$$

and

$$p^{2n+1}_{i,i} = b_i \left(\frac{x_i}{1 - x_i} - \frac{x_i}{x_{2n+1} - x_i} \right) = b_i \left(\frac{x_i}{1 - x_i} - \frac{x_i}{-1 - x_i} \right)$$

$$p^{2n+1}_{i+n,i+n} = b_{i+n} \left(\frac{x_{i+n}}{1 - x_{i+n}} - \frac{x_{i+n}}{x_{2n+1} - x_{i+n}} \right) = b_{i+n} \left(\frac{-x_i}{1 + x_i} - \frac{-x_i}{-1 + x_i} \right)$$

$$p^k_{i,i} + p^k_{i+n,i+n} = b_i \left(\frac{x_i}{1 - x_i} - \frac{x_i}{x_k - x_i} \right) + b_{i+n} \left(\frac{-x_i}{1 + x_i} - \frac{-x_i}{x_k + x_i} \right)$$

$$p^{k+n}_{i,i} + p^{k+n}_{i+n,i+n} = b_i \left(\frac{x_i}{1 - x_i} - \frac{x_i}{-x_k - x_i} \right) + b_{i+n} \left(\frac{-x_i}{1 + x_i} - \frac{-x_i}{-x_k + x_i} \right)$$

$$p^i_{i,i} + 2p^i_{i,i+n} + p^i_{i+n,i+n} = - \sum_{1 \leqslant k \neq i \leqslant 2n+1} b_k \frac{x_k}{x_i - x_k} + \frac{1}{1 - x_i} \{(a + b_i)x_i - c + 1\}$$

$$+2b_{i+n}\frac{x_{i+n}}{x_i-x_{i+n}}+b_{i+n}\left(\frac{x_{i+n}}{1-x_{i+n}}-\frac{x_{i+n}}{x_i-x_{i+n}}\right)$$
$$=-\sum_{1\leqslant k\neq i\leqslant n}\left(b_k\frac{x_k}{x_i-x_k}+b_{k+n}\frac{-x_k}{x_i+x_k}\right)$$
$$+\frac{(a+b_i+b_{i+n})x_i^2+(a-c-b_{2n+1}+1+b_i-b_{i+n})x_i+b_{2n+1}-c+1}{(1-x_i)(1+x_i)}$$

$$p^{i+n}_{i,i}+2p^{i+n}_{i,i+n}+p^{i+n}_{i+n,i+n}=b_1\left\{\frac{x_i}{1-x_i}-\frac{x_i}{-x_i-x_i}\right\}+2b_i\frac{x_i}{-x_i-x_i}$$
$$+\frac{1}{1+x_i}\left\{(a+b_{i+n})(-x_i)-c+1\frac{-1}{x_i}\right\}-\sum_{1\leqslant k\neq i+n\leqslant 2n+1}b_k\frac{x_k}{x_{i+n}-x_k}$$
$$=-\sum_{1\leqslant k\neq i\leqslant n}\left(b_k\frac{x_k}{-x_i-x_k}+b_{k+n}\frac{-x_k}{-x_i+x_k}\right)$$
$$+\frac{(a+b_i+b_{i+n})x_i^2+(-a+c+b_{2n+1}-1+b_i-b_{i+n})x_i+b_{2n+1}-c+1}{(1-x_i)(1+x_i)}.$$

In order for the system for v to be of rank $n+1$, D_iD_jv must be linearly related to D_kv and v. We have

$$p^{2n+1}_{i,i}+p^{2n+1}_{i+n,i+n}=0 \qquad\text{and}\qquad p^k_{i,j}+p^k_{i+n,j+n}=p^{k+n}_{i,j}+p^{k+n}_{i+n,j+n}\qquad(1\leqslant i,j\leqslant n)$$

if and only if $b_i=b_{i+n}$ $(1\leqslant i\leqslant n)$. Assuming these, we have

$$p^i_{i,i}+2p^i_{i,i+n}+p^i_{i+n,i+n}=p^{i+n}_{i,i}+2p^{i+n}_{i,i+n}+p^{i+n}_{i+n,i+n}\qquad(1\leqslant i\leqslant n)$$

if and only if $a-c-b_{2n+1}+1=0$.

Proposition 1. *The pull-back, under the embedding* $\iota: X(2,n+3)\to X(2,2n+4)$ *of a (any) non-zero solution of the hypergeometric system* $E_D^{2n+1}(a,b_1,\ldots,b_{2n+1},c)$ *satisfies a system of rank* $n+1$ *if and only if*

$$b_i=b_{i+n}\quad(1\leqslant i\leqslant n)\qquad\text{and}\qquad a-c-b_{2n+1}+1=0.$$

This system of rank $n+1$ *on* $X(2,n+3)$*, which will be called* $\mathcal{E}(a,b_1,\ldots,b_n,c)$*, is given by*

$$D_iD_jv=\sum_{k=1}^n q^k_{ij}D_kv+q^0_{ij}v\qquad(1\leqslant i,j\leqslant n)$$

with parameters $(a,b_1,\ldots,b_n,c)$*, where*

$$q^k_{ij}=q^k_{ji}\qquad q^0_{ij}=q^0_{ji}\qquad 1\leqslant i,j,k\leqslant n\qquad q^0_{ij}=q^k_{ij}=0\qquad i\neq j\neq k\neq i$$

$$q^i_{ij}=p^i_{i,j}+p^i_{i,j+n}=2b_j\frac{x_j^2}{x_i^2-x_j^2}\qquad i\neq j$$

$$q^0_{ii}=p^0_{i,i}+p^0_{i+n,i+n}=a(2b_i)\frac{x_i^2}{1-x_i^2}$$

$$q^k_{ii}=p^k_{i,i}+p^k_{i+n,i+n}=2b_i\left(\frac{x_i^2}{1-x_i^2}-\frac{x_i^2}{x_k^2-x_i^2}\right)\qquad i\neq k$$

$$q^i_{ii}=p^i_{i,i}+2p^i_{i,i+n}+p^i_{i+n,i+n}=-\sum_{k\neq i}2b_k\frac{x_k^2}{x_i^2-x_k^2}+\frac{(a+2b_i)x_i^2+a-2c+2}{1-x_i^2}.$$

The system has regular singularities along

$$\prod_{i=1}^{n} x_i(1-x_i)(1+x_i) \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)(x_i + x_j) = 0$$

and at infinity.

Rewriting this system in the form $\partial^2 v/\partial x_i \partial x_j = \cdots$, we find

Corollary 1. *This system is non-singular along* $\{x_i = 0\}$ *if and only if*

$$(q^i_{ii}|_{x_i=0} =)\ 2 \sum_{1\leqslant k(\neq i)\leqslant n} b_k + a - 2c + 2 = 0.$$

We need not check that this system is of rank $n+1$, because of the following fact. Note that if we introduce the variables $y_j = x_j^2$, then we have

$$D^x_j = 2D^y_j \qquad \text{where} \quad D^x_j = x_j \partial/\partial x_j \quad D^y_j = y_j \partial/\partial y_j.$$

Comparing the coefficients of $\mathcal{E}(a, b, c)$ and those of $E^n_D(a, b, c)$, we have

Corollary 2. *The Appell–Lauricella system* $E^n_D(a/2, b_1, \ldots, b_n, c - a/2)$ *in* y*-variables is transformed into* $\mathcal{E}(a, b, c)$ *in* x*-variables by the change* $y_j = x_j^2$.

Remark 1. The following integral representation of a solution of $E^{2n+1}_D(a, b, c)$

$$\int t^{a-1}(1-t)^{c-a-1}(1-x_1)^{-b_1} \cdots (1-x_{2n})^{-b_{2n}}(1-x_{2n+1})^{-b_{2n+1}}\, \mathrm{d}t$$

supports proposition 1 and corollary 2.

5. Systems invariant under the involution

Consider the involution

$$\# : (x_1, \ldots, x_n) \longmapsto (1/x_1, \ldots 1/x_n)$$

on $X(2, n+3)$ and let # take the system $E(a, b, c) = E^n_D(a, b_1, \ldots, b_n, c)$ into a system

$${}^{\#}E(a, b, c) : D_i D_j u = \sum_{k=1}^{n} \pi^k_{ij} D_k u + \pi^0_{ij} u \qquad (1 \leqslant i, j \leqslant n).$$

Since the change of variables $x_i \to 1/x_i$ induces the change of the derivations $D_i \to -D_i$, we can easily see the coefficients

$$\pi^k_{ij} = \pi^k_{ji} \qquad \pi^0_{ij} = \pi^0_{ji} \qquad 1 \leqslant i, j, k \leqslant n \qquad \pi^0_{ij} = \pi^k_{ij} = 0 \qquad i \neq j \neq k \neq i$$

$$\pi^i_{ij} = b_j \frac{x_i}{x_i - x_j} \qquad i \neq j \qquad \pi^0_{ii} = ab_i \frac{1}{1 - x_i}$$

$$\pi^k_{ii} = b_i \left(\frac{1}{1-x_i} - \frac{x_k}{x_k - x_i} \right) \qquad i \neq k$$

$$\pi^i_{ii} = -\sum_{k\neq i} b_k \frac{x_i}{x_i - x_k} + \frac{(-c+1)x_i + a + b_i}{1 - x_i}.$$

In particular, $E(a, b, c)$ is not invariant under the involution # for any choice of parameters not simultaneously zero.

Let us transform the systems E and ${}^{\#}E$ into the normal forms defined below. We change the unknown v of the system $E(a, b, c)$ as $v = \rho w$, where ρ is a non-zero function. If we write the new system as

$$ {}^{N}E(a,b,c) : D_i D_j w = \sum_{k=1}^{n} P_{ij}^{k} D_k w + P_{ij}^{0} w \qquad (1 \leqslant i, j \leqslant n) $$

then the coefficients P_{ij}^k are given by

$$ P_{ij}^{k} = p_{ij}^{k} - \frac{D_i \rho}{\rho}\delta_j^k - \frac{D_j \rho}{\rho}\delta_i^k $$

where δ denotes the Kronecker symbol. Now choose ρ so that the system ${}^{N}E$ is of *normal form*, which means by definition,

$$ \sum_{k=1}^{n} P_{kj}^{k} = 0 \qquad (1 \leqslant j \leqslant n) $$

that is,

$$ \sum_{k=1}^{n} p_{kj}^{k} - (n+1)\frac{D_j \rho}{\rho} = 0 \qquad (1 \leqslant j \leqslant n) $$

it is known that if the given system E is integrable then there is a non-zero function ρ solving the above first-order system of differential equations. Thus we have

$$ P_{ij}^{k} = p_{ij}^{k} - \frac{\delta_j^k}{n+1} p_i - \frac{\delta_i^k}{n+1} p_j $$

where

$$ p_j = \sum_{k=1}^{n} p_{kj}^{k} = \sum_{k \neq j} p_{kj}^{k} + p_{jj}^{j} = \sum_{k \neq j} \frac{b_j x_j + b_k x_k}{x_k - x_j} + \frac{(a+b_j)x_j - c + 1}{1 - x_j}. $$

The coefficients of the normal form ${}^{N}E(a, b, c)$ are given by

$$ P_{ii}^{k} = p_{ii}^{k} = \frac{b_i x_i (x_k - 1)}{(1 - x_i)(x_k - x_i)} \qquad i \neq k $$

$$ P_{ij}^{i} = p_{ij}^{i} - \frac{p_j}{n+1} = \frac{b_j x_j}{x_i - x_j} - \frac{p_j}{n+1} \qquad i \neq j $$

$$ P_{ii}^{i} = p_{ii}^{i} - 2\frac{p_j}{n+1}. $$

We next find the normal form

$$ {}^{N\#}E(a,b,c) : D_i D_j w = \sum_{k=1}^{n} \Pi_{ij}^{k} D_k w + \Pi_{ij}^{0} w \qquad (1 \leqslant i, j \leqslant n) $$

of ${}^{\#}E(a, b, c)$. Its coefficients are given by

$$ \Pi_{ij}^{k} = \pi_{ij}^{k} - \frac{\delta_j^k}{n+1}\pi_i - \frac{\delta_i^k}{n+1}\pi_j $$

where

$$ \pi_j = \sum_{k=1}^{n} \pi_{kj}^{k} = \sum_{k \neq j} \pi_{kj}^{k} + \pi_{jj}^{j} = \sum_{k \neq j} \frac{b_j x_k + b_k x_j}{x_k - x_j} + \frac{(-c+1)x_j + a + b_j}{1 - x_j}. $$

Thus we have

$$\Pi^k_{ii} = \pi^k_{ii} = \frac{b_i x_i (x_k - 1)}{(1 - x_i)(x_k - x_i)} \qquad i \neq k$$

$$\Pi^i_{ij} = \pi^i_{ij} - \frac{\pi_j}{n+1} = \frac{b_j x_i}{x_i - x_j} - \frac{\pi_j}{n+1} \qquad i \neq j$$

$$\Pi^i_{ii} = \pi^i_{ii} - 2\frac{\pi_j}{n+1}.$$

Compare the coefficients and recall the elementary fact that a rational function

$$\sum_j \frac{\alpha_j x + \beta_j}{x - t_j}$$

in x vanishes identically if and only if $\alpha_j t_j + \beta_j = 0$ and $\sum_j \alpha_j = 0$. Then we find the following.

Proposition 2. *The system* ${}^N E(a, b, c)$ *coincides with the system* ${}^{N\#}E(a, b, c)$ *if and only if*

$$b_1 = \cdots = b_n (= b) \qquad -nb + a + c - 1 = 0.$$

Let us summarize the above computation. We obtained a system $\mathcal{E}(a, b, c) = \mathcal{E}(a, b_1, \ldots, b_n, c)$ of rank $n+1$ on $X(2, n+3)$, which was the pull-back under $\iota : X(2, n+3) \to X(2, 2n+4)$ of the hypergeometric system $E_D^{2n+1}(a, b_1, \ldots, b_{2n+1}, c)$, with

$$b_{n+1} = b_1, \ldots, b_{2n} = b_n \qquad b_{2n+1} = a - c + 1.$$

The normal forms of $\mathcal{E}(a, b, c)$ and its pull-back under # coincide if and only if

$$b_1 = \cdots = b_n \ (= b) \qquad -(2n+1)b + a + c - 1.$$

These conditions lead to

$$a = (n+1)b \qquad c = nb + 1.$$

If moreover $\mathcal{E}$ is regular along the divisors $\{x_j = 0\}$, then we have $b = 1/(n-1)$.

6. The double covering $f : X(2, 7) \to X(3, 6)$

Let $X(3, 6)$ be the configuration space of coloured six points in general position in the projective plane

$$X(3, 6) = GL_3 \backslash \{z \in M(3, 6) | \text{no 3-minor of } z \text{ vanishes}\}/(\mathbf{C}^\times)^6.$$

We define a rational map f from $X(2, 7)$ to $X(3, 6)$. We start from a system of seven points on the line representing a point of $X(2, 7)$. We regard the line, carrying the seven points, as a non-singular conic in the plane. The five points represented by the last five points, and the intersection point of the tangent lines (to the conic) at the first and the second points define a system of six points on the plane, representing a point of $X(3, 6)$. Let us express this map f in terms of coordinates. We normalize the system of seven points to be

$$x = (0, \infty, 1, x_1, \ldots, x_4) \in X(2, 7).$$

If the conic is given by $t_1^2 - t_0 t_2 = 0$ in the plane coordinatized by $t_0 : t_1 : t_2$, the seven points are represented by the seven columns

$$\begin{matrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & x_1 & x_2 & x_3 & x_4 \\ 0 & 1 & 1 & x_1^2 & x_2^2 & x_3^2 & x_4^2. \end{matrix}$$

Since the tangents of the conic at the first two points are $t_2 = 0$ and $t_0 = 0$, the intersection is given by $0 : 1 : 0$; so the point $f(x)$ is represented by

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & x_1 & x_2 & x_3 & x_4 \\ 0 & 1 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{pmatrix} \in X(3,6).$$

Normalizing the above 3×6-matrix into

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & z_1 & z_2 \\ 0 & 0 & 1 & 1 & z_3 & z_4 \end{pmatrix}$$

we can readily show

Proposition 3. *The map*

$$f : X(2,7) \ni x = (x_1, \dots, x_4) \longmapsto z = (z_1, \dots, z_4) \in X(3,6)$$

defined in this section is given by

$$z_1 = \frac{(x_3 + x_1)(x_2 - 1)}{(x_3 - 1)(x_2 + x_1)} \qquad z_2 = \frac{(x_4 + x_1)(x_2 - 1)}{(x_4 - 1)(x_2 + x_1)}$$

$$z_3 = \frac{(x_3 + 1)(x_2 - x_1)}{(x_3 - x_1)(x_2 + 1)} \qquad z_4 = \frac{(x_4 + 1)(x_2 - x_1)}{(x_4 - x_1)(x_2 + 1)}.$$

Remark 2.

(a) The Jacobian of f is given by

$$\frac{(x_1 - 1)(-x_2 + x_1)(x_2 - x_4)(x_3 - x_4)(x_3 - x_2)(x_1 + 1)^3(x_2 - 1)}{(x_3 - x_1)^2(x_2 + 1)^3(x_4 - 1)^2(x_2 + x_1)^3(x_3 - 1)^2(-x_4 + x_1)^2}.$$

(b) Put

$$D_1 = z_1 z_4 - z_2 z_3 \qquad D_2 = z_1 z_4 - z_2 z_3 - z_4 + z_2 + z_3 - z_1$$

$$Q = -z_2 z_3 z_1 - z_2 z_3 z_4 + z_2 z_3 + z_1 z_4 z_2 + z_1 z_4 z_3 - z_1 z_4.$$

Then the singular locus of the system found in [3] is defined by

$$D = \prod_{j=1}^{4} z_j(1 - z_j) \cdot (z_1 - z_2)(z_1 - z_3)(z_2 - z_4)(z_3 - z_4) D_1 D_2 Q$$

and $f^*(D)$ is given by

$$\begin{aligned} &-(x_3 + x_1)(x_2 - 1)^5(x_4 + x_1)(x_3 + 1)(-x_2 + x_1)^5(x_4 + 1)(x_1 + 1)^{13} \\ &\quad \times (x_3 - x_2)^5(x_2 - x_4)^5(x_3 - x_4)^5(x_1 - 1)^5(x_3 + x_2)(x_2 + x_4)(x_3 + x_4) \\ &\quad \times (x_3 - 1)^{-7}(x_2 + x_1)^{-11}(x_4 - 1)^{-7}(x_3 - x_1)^{-7}(x_2 + 1)^{-11}(-x_4 + x_1)^{-7}. \end{aligned}$$

Proposition 4. *The map f is invariant under the involution $\# : x_i \to 1/x_i$. Moreover, f is a two-to-one map.*

Sketch of the proof. For given $(z_1, \dots, z_4)$, we solve $x_1, \dots, x_4$. We can see that x_2 must be a solution of the quadratic equation

$$C_2 Z^2 + C_1 Z + C_0 = 0$$

where

$$\begin{aligned}
C_0 = C_2 = {} & -(-z_2z_4 + z_3z_2z_4 - z_3 + z_1z_3 + z_4 - z_1z_3z_4) \\
& \times(z_3 + z_2 - z_4 - z_1 - z_1z_3z_4 + z_1z_3z_2 - z_2z_4z_1 + z_3z_2z_4 - 2z_2z_3 + 2z_1z_4) \\
C_1 = {} & -6z_1z_3z_4 + 4z_2^2z_4z_1 - 4z_2^2z_1z_3 + 4z_3^2z_2^2z_1 + 4z_4^2z_1^2z_2 - 4z_1^2z_2z_4 + 4z_1^2z_3z_2 \\
& -2z_1z_3z_2 - 2z_2z_4z_1 - 6z_3z_2z_4 - 6z_2z_4^2z_1z_3 + 16z_2z_4z_1z_3 - 2z_1^2z_3z_4^2z_2 \\
& -2z_3^2z_1z_2 - 2z_1^2z_3^2z_2 - 2z_4^2z_2z_1 - 2z_1^2z_3z_4 + 2z_1^2z_3^2z_4 - 2z_3^2z_2^2z_4^2 - 2z_3z_2^2z_4 \\
& -2z_2^2z_4^2z_1 + 2z_2^2z_4^2z_3 + 4z_3^2z_2^2z_4 - 2z_1^2z_3^2z_4^2 + 4z_1^2z_3z_4^2 - 2z_3^2 - 2z_4^2 \\
& -4z_3^2z_2^2 + 2z_1z_3 - 2z_2z_3 + 4z_4z_3 + 4z_4z_1^2 - 4z_4^2z_1^2 + 2z_2z_4 + 4z_1z_4^2 - 2z_1z_4 \\
& +2z_1z_3^2 - 2z_1^2z_3 + 4z_3^2z_2 - 2z_2^2z_4 + 2z_2z_4^2 + 2z_1^2z_3^2z_4z_2 + 4z_3z_2^2 + 2z_3z_2^2z_4^2z_1 \\
& -2z_3^2z_2^2z_4z_1 - 2z_2^2z_4z_1z_3 + 4z_1z_3^2z_4^2z_2 - 2z_1^2z_3z_2z_4 - 6z_1z_3^2z_2z_4.
\end{aligned}$$

The discriminant of the quadratic equation with respect to Z is given by

$$16(z_1 - 1)(z_2 - 1)(z_3 - 1)(z_4 - 1)(z_1 - z_2)(z_3 - z_4)D_2Q.$$

The other quantities x_1 and x_3 can be expressed as rational functions in x_2 and z:

$$\begin{aligned}
x_1 = {} & -x_2(-z_1z_3 + z_1z_3z_2 + z_2z_4 - z_2z_4z_1 - z_2 + z_1)/(-z_3x_2 + z_3 + z_2 + z_4x_2 \\
& -z_4 - z_1 - z_1z_3z_4 - z_1z_3z_4x_2 + z_1z_3z_2 - z_2z_4z_1 + z_3z_2z_4x_2 + z_3z_2z_4 \\
& +x_2z_1z_3 - 2z_2z_3 - x_2z_2z_4 + 2z_1z_4) \\
x_3 = {} & -x_2(1 - 2z_1 + z_1z_3 - x_2 + x_2z_1z_3)(z_3 + z_2 - z_4 - z_1 - z_1z_3z_4 + z_1z_3z_2 \\
& -z_2z_4z_1 + z_3z_2z_4 - 2z_2z_3 + 2z_1z_4)/(2x_2z_3^2z_1z_4 + z_3 + z_2 - z_4 - z_1 \\
& -6z_1z_3z_4 + 2z_1z_3z_2 - z_2z_4z_1 + z_3z_2z_4 + 2z_2z_4z_1z_3 - 4z_3^2z_1z_2 \\
& +z_1^2z_3^2z_2 + 2z_1^2z_3z_4 - z_1^2z_3^2z_4 - 2z_3^2 + 2z_1z_3 - 4z_2z_3 + 2z_4z_3 \\
& +2z_1z_4 + z_1z_3^2 - z_1^2z_3 + 4z_3^2z_2 - z_1^2z_3z_2z_4 + z_1z_3^2z_2z_4 \\
& -2x_2z_3^2z_2z_4 - 2x_2z_1^2 - 2x_2z_1^2z_3z_2 + 2x_2z_3^2 + 3x_2z_1^2z_3 - x_2z_2 \\
& +x_2z_1 - 3x_2z_2z_4z_1 + x_2z_1z_3^2z_2z_4 - 2x_2z_4z_3 - 2z_3^2z_2z \\
& +2z_1z_3^2z_4 - z_3x_2 + z_4x_2 + 2x_2z_1z_2 + 2x_2z_1^2z_2z_4 + 3z_3z_2z_4x_2 \\
& -x_2z_1^2z_3z_2z_4 - x_2z_1^2z_3^2z_4 - 3x_2z_1z_3^2 + x_2z_1^2z_3^2z_2)
\end{aligned}$$

and x_4 is obtained from the expression of x_3 by the exchanges $z_1 \leftrightarrow z_2$ and $z_3 \leftrightarrow z_4$. Therefore, the map f is two-to-one. □

7. A system on $X(3, 6)$ induced by $\mathcal{E}$ invariant under

We now have the system

$$\mathcal{E}(a, b, c) \qquad a = 5b \quad c = 4b + 1 \quad b = b_1 = \cdots = b_4$$

with a parameter b on $X(2, 7)$ invariant under the involution #. The push-down $f_*\mathcal{E}(a, b, c)$ of this system is a system defined on $X(3, 6)$. Here we confess honestly that we still have not found a way to express the system $f_*\mathcal{E}(a, b, c)$ in z-variables with parameter b in a reasonably compact form. Nevertheless, when the system $\mathcal{E}(a, b, c)$ is non-singular along the divisors $\{x_j = 0\}$, that is, when $b = \frac{1}{3}$, we can explicitly find the coefficients of the system $f_*\mathcal{E}(a, b, c)$ and we obtain

Theorem 1. *The system* $\mathcal{E}(\frac{5}{3}, \frac{1}{3}, \frac{7}{3})$*, as a system on* $X(3, 6)$*, coincides with the system we found in [3], which has singularities along the divisor* $\{D = 0\}$.

Make the coordinate change $x \to z$ to transform the system $\mathcal{E}(a, b, c)$ into the form $\partial^2 v/\partial z_i \partial z_j = \cdots$ and write the coefficients in terms of x. Though it is possible to express these coefficients in z, it is much easier to rewrite the *coefficients* of the system in [3] in terms of x. We then compare the coefficients of the two systems to find that they coincide.

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